ON A CRITERION OF PRINGSHEIM'S FOR EXPANSIBILITY IN TAYLOR'S SERIES

By M. B. Porter

In his paper on Taylor's series* Pringsheim has established, among others, the following criterion for the expansibility of a function in Taylor's series:

If we denote by C_n the Cauchy Remainder:

$$C_n = h^n f^{(n)}(x + \theta h) \frac{(1-\theta)^{n-1}}{(n-1)!},$$

then *the necessary and sufficient condition that*

$$f(x+h) = \sum_0^\infty f^{(n)}(x) \frac{h^n}{n!} \qquad (0 \leqq h < R)$$

is that

$$\lim_{n=\infty} C_n = 0 \tag{1}$$

uniformly for all values of θ and h such that $0 \leqq \theta \leqq 1$, $0 \leqq h \leqq R_1 < R$; or, as we may say, over the "rectangle" $[0, R_1;\ 0, 1]$.

The sufficiency of Pringsheim's criterion is obvious at once; the proof of its necessity is a more delicate matter. The following considerations furnish a simple and elementary proof of Pringsheim's criterion which is applicable without modification to the case when f is a *complex* function of a real variable.

* Zum Taylor'schen Lehrsatz, *Math. Ann.*, vol. 44 (1894), p. 68.

If we integrate the right-hand side of the identity

$$f(x+h)-f(x)=\int_0^h f'(x+h-t)\,dt$$

$n-1$ timos by parts we get

$$f(x+h)=\sum_{m=0}^{n-1} f^{(m)}(x)\frac{h^m}{m!}+\int_0^h f^{(n)}(x+h-t)\ \frac{t^{n-1}}{(n-1)!}\,dt,$$

where the integral on the right is Lagrange's integral for the remainder. In this integral set

$$h-t=\theta h$$

and it becomes

$$R_n=h^n\int_0^1 f^{(n)}(x+\theta h)\,\frac{(1-\theta)^{n-1}}{(n-1)!}\,d\theta.$$

Thus, *the necessary and sufficient condition that*

$$\sum_0^\infty f^{(n)}(x)\frac{h^n}{n!}$$

converge to the limit $f(x+h)$ *is that*

$$\lim_{n=\infty} R_n=0.$$

Since a power series always converges uniformly over any interval lying *inside* its interval of convergence we have that

$$\lim_{n=\infty} R_n=0 \tag{2}$$

uniformly for every h in the interval

$$0\leqq h\leqq R_1<R. \tag{3}$$

Now for (2) to have place it is obviously *sufficient* that the integrand in R_n, i, e., C_n, should uniformly vanish over the rectangle $[0, R_1;\ 0, 1]$. To prove that this condition is also *necessary* we proceed as follows. Since by hypothesis

$$f(x+h)=\sum_0^\infty f^{(n)}(x)\frac{h^n}{n!} \qquad (0\leqq h<R),$$

differentiating p times with respect to h we get

$$f^{(p)}(x+h)=\sum_0^\infty f^{(n+p)}(x)\frac{h^n}{n!} \qquad (0\leqq h<R);$$

i. e., if $f(x+h)$ is expansible by Taylor's series, so is $f^{(p)}(x+h)$. Thus taking $p=1$ we have

$$\lim_{n=\infty}\int_0^1 h^n f^{(n+1)}(x+\theta h)\frac{(1-\theta)^{n-1}}{(n-1)!}d\theta=0$$

uniformly for every $0\leqq h\leqq R_1<R$.

The fundamental lemma on which our proof rests is the following:

LEMMA. *If*

$$\lim_{n=\infty}\int_0^1 h^n f^{(n+1)}(x+\theta h)\frac{(1-\theta)^{n-1}}{(n-1)!}d\theta=0$$

uniformly over $0\leqq h\leqq R_1$, *then*

$$\lim_{n=\infty}\int_\theta^1 \left| h^n f^{(n+1)}(x+\theta h)\frac{(1-\theta)^{n-1}}{(n-1)!}\right| d\theta=0$$

uniformly over $[0, R_1;\ 0, 1]$.

To prove this lemma it will evidently suffice to show that

$$\lim_{n=\infty}\int_0^1 \left| h^n f^{(n+1)}(x+\theta h)\frac{(1-\theta)^{n-1}}{(n-1)!}\right| d\theta=0$$

uniformly over $0\leqq h\leqq R_1$.

Replacing $f^{(n+1)}(x+\theta h)$ by $\sum_{p=0}^{\infty} f^{(n+p+1)}(x)\frac{(\theta h)^p}{p!}$ (its expansion by Taylor's series) we get

$$\begin{aligned}\int_0^1 &\left| h^n f^{(n+1)}(x+\theta h)\frac{(1-\theta)^{n-1}}{(n-1)!}\right| d\theta\\ &\leqq \int_0^1 h^n\sum_{p=0}^{\infty}\left|f^{(n+p+1)}(x)\right|\frac{(\theta h)^p}{p!}\,\frac{(1-\theta)^{n-1}}{(n-1)!}\,d\theta\\ &\leqq \sum_{p=0}^{\infty}\left|f^{(n+p+1)}(x)\right|\frac{h^{n+p}}{(n+p)!}.^*\end{aligned}$$

This last expression is the sum of the absolute values of the terms after the n^{th} in the Taylor's expansion of $f'(x+h)$, hence it vanishes uniformly over $0\leqq h\leqq R_1$ when n becomes infinite, and the lemma is proved.

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* We use here the well known formula

$$\int_0^1\theta^p(1-\theta)^{n-1}d\theta=\frac{p!(n-1)!}{(n+p)!}.$$

To complete the proof, integrating by parts we have

$$h\int_\theta^1 h^n f^{(n+1)}(x+\theta h)\,\frac{(1-\theta)^{n-1}}{(n-1)!}\,d\theta = -\,h^n f^{(n)}(x+\theta h)\,\frac{(1-\theta)^{n-1}}{(n-1)!}$$
$$-\,h\int_\theta^1 h^{n-1} f^{(n)}(x+\theta h)\,\frac{(1-\theta)^{n-2}}{(n-2)!}\,d\theta,$$

and since by the above lemma both of these integrals vanish uniformly over $[0, R_1;\ 0, 1]$, then so must

$$h^n f^n(x+\theta h)\,\frac{(1-\theta)^{n-1}}{(n-1)!},$$

which proves the necessity of the criterion.

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